

Program : **B.Tech** Subject Name: **Signals and Systems** Subject Code: **EC-402** Semester: **4th**





Unit-3 z-Transform: Introduction, ROC of finite duration sequence, ROC of infinite duration sequence, Relation between Discrete time Fourier Transform and z-transform, properties of the ROC, Properties of z-transform, Inverse z-Transform, Analysis of discrete time LTI system using z-Transform, Unilateral z-transform.

Introduction:

The z-transform of x(n) is denoted by X(z). It is defined as,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

<u>Region of Convergence (ROC)</u>: The set of values of z in the z-plane for which the magnitude of X(z) is finite is called the Region of Convergence (ROC). The ROC of X(z) consists of a circle in the z-plane centered about the origin. The Fig.3.1 shows the two possible representation of ROC in z-plane.



Fig.3.1 : Region of Convergence Plot in Z-Plane

Significance of ROC:

(i) ROC gives an idea about values of z for which z-transform can be calculated

(ii) ROC can be used to test the causality of the system.

(iii) ROC can also be used to test the stability of the system.

Examples: (1) Determine the z-transform of following sequence



(i)
$$x_1(n) = \{1, 2, 3, 4, 5, 0, 7\}$$
 (ii) $x_2(n) = \{1, 2, 3, 4, 5, 0, 7\}$

Solution: (i) Given that $x_1(n) = \{1, 2, 3, 4, 5, 0, 7\}$ By definition,

$$X_{1}(z) = \sum_{n=-\infty}^{\infty} x_{1}[n]z^{-n} \qquad \therefore X_{1}(z) = \sum_{n=0}^{6} x_{1}[n]z^{-n}$$
$$\therefore X_{1}(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + x(5)z^{-5} + x(6)z^{-6}$$
$$X_{1}(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4} + 7z^{-6}$$

 $X_1(z)$ is convergent for all values of z, except z=0. Because $X_1(z) = \infty$ for z=0. Therefore the ROC is entire z=plane except z=0.

(ii) Given that $x_2(n) = \{1, 2, 3, 4, 5, 0, 7\}$ By definition,

$$X_{2}(z) = \sum_{n=-\infty}^{\infty} x_{2}[n]z^{-n} \qquad \therefore X_{2}(z) = \sum_{n=-3}^{3} x_{2}[n]z^{-n}$$
$$\frac{\therefore X_{2}(z) = x(-3)z^{3} + x(-2)z^{2} + x(-1)z^{1} + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}}{X_{1}(z) = z^{3} + 2z^{2} + 3z^{1} + 4 + 5z^{-1} + 7z^{-3}}$$

 $X_2(z)$ is convergent for all values of z, except z=0 and z= ∞ . Because $X_2(z) = \infty$ for z=0. Therefore the ROC is entire z=plane except z=0 and z= ∞ .



Z-Transform of Unit Step, u(n)
We know that u(n) = 1 for n
$$\geq 0$$

= 0 otherwise
By definition

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$
Now $x(n) = u(n)$ present from n=0 to ∞
 $\therefore X(z) = \sum_{n=0}^{\infty} u(n)z^{-n}$
 $\therefore X(z) = \sum_{n=0}^{\infty} 1.z^{-n}$
= $1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + \dots$
 $X(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$
The above equation converges if $|z^{-1}| < 1$ i.e., ROC is
 $|z| > 1$. Therefore the ROC is the exterior to the unit
circle in the z-plane.

$$\sum_{n=-\infty}^{\infty} x(n)z^{-n} = -\sum_{n=1}^{\infty} (a^{-1}z)^{-n} = -\sum_{n=1}^{\infty} (a^{-1}z)^{n} = -\sum_{n=1}^{\infty} (a^{-1}z)^{n} + 1 = -[1 + (a^{-1}z)^{1} + (a^{-1}z)^{2} + (a^{-1}z)^{3} + \dots] + 1$$

Relation between Discrete time Fourier Transform and z-transform

The Z-transform of a discrete sequence x(n) is defined The Fourier transform of a discrete sequence x(n) is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad \qquad X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

The X(z) is the unique representation of the sequence x(n) in the complex z-plane. Let $z = re^{j\omega}$

$$\therefore X(z) = \sum_{n=-\infty}^{\infty} x(n) (re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} [x(n)r^{-n}]e^{-j\omega n}$$

The RHS of the above equation is the Fourier transform of $x(n)r^{-n}$, \therefore The Z-transform of x(n) is the Fourier transform of $x(n)r^{-n}$. If r=1, then



$$\therefore X(z) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = X(\omega)$$

Therefore the Fourier transform of x(n) is same as the Z-transform of x(n) evaluated along the unit circle centered at the origin of the z-plane.

$$\therefore X(\omega) = X(z)|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

For $X(\omega)$ to exist, the ROC must include the unit circle. Since ROC cannot contain any poles of X(z) all the poles must lie inside the unit circle. Therefore, the Fourier transform can be obtained from Z-transform X(z) for any sequence x(n) if the poles of X(z) are inside the unit circle.

Properties of Z – Transform:

The Z-transform has different properties which can be used to obtain the z-transform of a given sequence. Any complex sequence z-transform can be determined by using the properties, which makes the z-transform a powerful tool for discrete-time system analysis.

(i) Linearity Property	(ii) Time Shifting Property	
It states that, the Z-transform of a weighted sum of two	It states that,	
sequences is equal to the weighted sum of individual Z-	$x(n) \xleftarrow{ZT} X(z)$ with $BOC = B$	
transforms.	$\prod_{i=1}^{n} x_{i}(i) = x_{i}(i) + x_{i}(i)$	
If $x_1(n) \xleftarrow{ZT} X_1(z)$, with ROC = R_1	then $x(n-m) \xleftarrow{21}{\longleftarrow} z^{-m} X_2(z)$,	
and $x_2(n) \xleftarrow{2T} X_2(z)$, with ROC = R_2	with ROC =R except for the possible addition or	
then	deletion of the origin or infinity.	
$ax_1(n) + bx_2(n) \leftrightarrow aX_1(z) + bX_2(z)$, with $ROL = R_1 \cap R_2$	<u><i>Proof:</i></u> By definition	
	\sim	
<u>Proof:</u> By definition	$Z[x(n)] = X(z) = \sum x(n)z^{-n}$	
$Z[x(n)] - X(z) - \sum_{n=1}^{\infty} x(n) z^{-n}$	$n=-\infty$	
$\Sigma[X(\Pi)] = X(\Sigma) = \sum_{n=1}^{\infty} X(\Pi)\Sigma$	\sim	
$n = -\infty$	$Z[x(n-m)] = \sum x(n-m)z^{-n}$	
$\sum_{n=1}^{\infty} [a_n, (n)] = \sum_{n=1}^{\infty} [a_n, (n)] + b_n, (n)] = 0$	$n=-\infty$	
$Z[ax_1(n) + bx_2(n)] = \sum_{n=1}^{\infty} [ax_1(n) + bx_2(n)]z^{n}$	Put p=n-m in the summation, then n=m+p.	
$n=-\infty$	$\sum_{n=1}^{\infty} (n+n)$	
$\sum_{n=1}^{\infty}$ () $\sum_{n=1}^{\infty}$ () $\sum_{n=1}^{\infty}$	$Z[x(n-m)] = \sum x(p)z^{-(m+p)}$	
$= \sum_{n=1}^{\infty} ax_1(n)z^{-n} + \sum_{n=1}^{\infty} bx_2(n)z^{-n}$	$p = -\infty$	
$n = -\infty$ $n = -\infty$	$7[y(n - m)] = \pi^{-m} \sum y(n)\pi^{-p}$	
\sim \sim \sim	$\Sigma[x(n-m)] = \Sigma \qquad \sum x(p)\Sigma$	
$= a \sum_{n=1}^{\infty} x_1(n)z^{-n} + b \sum_{n=1}^{\infty} x_2(n)z^{-n}$	p=−∞	
$n=-\infty$ $n=-\infty$	$Z[x(n-m)] = z^{-m}X(z)$	
$= aX_1(z) + bX_2(z); ROC = R_1 \cap R_2$		
	$Z[x(n-m)] \leftrightarrow z^{-m}X(z)$	
$ax_1(n) + bx_2(n) \stackrel{\sim}{\leftrightarrow} aX_1(z) + bX_2(z)$	ZT $7[y(n+m)] \leftrightarrow \pi^{m}Y(\pi)$	



(iii) Multiplication by an Exponential Sequence Property	(iv) Time Reversal Property
(Scaling Property)	It states that,
It states that,	If $x(n) \stackrel{ZT}{\leftrightarrow} X(z)$, with ROC = R
If $x(n) \stackrel{Z_1}{\leftrightarrow} X(z)$, with ROC = R	then $x(-n) \stackrel{ZT}{\leftrightarrow} X(\frac{1}{2})$, with $ROC = \frac{1}{2}$
then $a^{n}x(n) \stackrel{ZT}{\leftrightarrow} X\left(\frac{z}{a}\right)$, with ROC = $ a R$	(z), (z) , (z) , R
where 'a' is a complex number	<u><i>Proof:</i></u> By definition
	$Z[x(n)] = X(z) = \sum_{n=1}^{\infty} x(n)z^{-n}$
<u><i>Proof:</i></u> By definition	$n=-\infty$
$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$	$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)z^{n}$
$Z[a^{n}x(n)] = \sum_{n=-\infty}^{\infty} a^{n}x(n)z^{-n}$	$Z[x(-n)] = \sum_{n=1}^{\infty} x(-n)(z^{-1})^{-n}$
$=\sum_{n=-\infty}^{\infty} x(n) \left(\frac{z}{a}\right)^{-n}$	$\therefore \mathbf{Z}[\mathbf{x}(-\mathbf{n})] = \mathbf{X}(\mathbf{z}^{-1})$
$= X\left(\frac{z}{a}\right)$	
$a^n x(n) \stackrel{ZT}{\leftrightarrow} X\left(\frac{z}{a}\right)$.IN
$e^{j\omega n}x(n) \stackrel{ZT}{\leftrightarrow} X\left(\frac{z}{e^{j\omega}}\right) = X(e^{-j\omega}z)$	
$e^{-j\omega n}x(n) \stackrel{ZT}{\leftrightarrow} X\left(\frac{z}{e^{-j\omega}}\right) = X(e^{j\omega}z)$	

SI. No.	Time Domain Sequence x(n)	Z-Transform X(z)	ROC
1.	δ(n)	1	Entire Z-plane
2.	u(n)	$\frac{1}{1-z^{-1}}$	z >1
3.	a ⁿ u(n)	$\frac{1}{1-az^{-1}}$	z > a
4.	-a ⁿ u(-n-1)	$\frac{1}{1-az^{-1}}$	z < a
5.	n.a ⁿ u(n)	$\frac{az^{-1}}{(1-az^{-1})^2}$	z > a
6.	-n.a ⁿ u(-n-1)	$\frac{az^{-1}}{(1-az^{-1})^2}$	z < a



- (i) Power Series Expansion (Long Division Method)
- (ii) Partial Fraction Expansion
- (iii) Contour Integration (Residue Method)

(i) Power Series Expansion (Long Division Method):

Example: Determine the inverse z-transform of the following (i) $X(z) = (1/(1-az^{-1}), ROC |z| > |a|$ (ii) $X(z) = (1/(1-az^{-1}), ROC |z| < |a|$

Solution: (i)

$$1+az^{-1}+a^{2}z^{-2}+a^{3}z^{-3}+-----$$

$$1-az^{-1}) 1 1 - az^{-1} - a^{2}z^{-2} - a^{2}z^{-2} - a^{2}z^{-2} - a^{3}z^{-3} - \frac{-a^{2}z^{-2}}{a^{3}z^{-2}-a^{3}z^{-3}} - \frac{-a^{2}z^{-2}}{a^{3}z^{-3}-a^{4}z^{-4}} - \frac{-a^{2}z^{-4}}{a^{4}z^{-4}-----}$$

 $a^{-3}z^3 - a^{-4}z^4$

a⁻⁴7⁻⁴ -----

Thus we have

$$X(z) = \frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^{2}z^{-2} + a^{3}z^{-3} + \dots - \dots$$

Taking Inverse z-transform , $x(n) = \{1, a, a^{2}, a^{3}, \dots, \dots\}$
 $x(n) = a^{n} u(n)$
Solution: (ii)
 $az^{-1} + 1$
 $az^{-1}z$
 $a^{-1}z$
 $a^{-1}z - a^{-2}z^{2}$
 $a^{-1}z - a^{-2}z^{2}$
 $a^{-1}z - a^{-2}z^{2}$
 $a^{-2}z^{2} - a^{-3}z^{3}$
 $a^{-2}z^{2} - a^{-3}z^{3}$
 $a^{-2}z^{2} - a^{-3}z^{3}$

$$\begin{split} X(z) &= \frac{1}{1-az^{-1}} = -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 + - - - \\ \text{Faking Inverse z-transform ,} \quad x(n) = \{----, -a^{-3}, -a^{-2}, -a^{-1}, \} \\ &\qquad x(n) = -a^n \ u(-n-1) \end{split}$$





(ii) Partial Fraction Method:

Step - 1 : First convert given X(z) into positive powers of z and then write $\frac{X(z)}{z}$

- Step 2 : Using partial fraction method, write the equation in terms of summation of poles. Find the constants in the numerator.
- Step 3 : Rewrite the equation in the form of X(z).
- Step 4 : Based on the condition of ROC, write the inverse z-transform x(n) of X(z).

(iii) Contour Integration (Residue Method:

Step-1: Define the function $X_0(z)$ which is rational and its denominator is expanded into product of poles. $X_0(z) = X(z) z^{n-1}$

Step-2: (i) For Simple poles, the residue of $X_0(z)$ at pole p_i is given as,

$$\operatorname{Res}_{z=p_i} = \lim_{z=p_i} (z - p_i) X_0(z)$$

(ii) For multiple poles of order m_0 , the residue of $X_0(z)$ can be calculated as,

$$\underset{z=p_{i}}{\text{Res}} = \frac{1}{(m-1)!} \left\{ \frac{d^{m-1}}{dz^{m-1}} (z-p_{i}) X_{0}(z) \right\}_{z=p_{i}}$$

Step-3: (i) Using residue theorem, calculate x(n), for poles inside the unit circle

$$x(n) = \sum_{i=1}^{N} \underset{z=p_i}{\text{Res}} X_0(z) \text{ with } n \ge 0$$

(ii) Using residue theorem, calculate x(n), for poles outside the unit circle

$$x(n) = -\sum_{i=1}^{N} \underset{z=p_i}{\text{Res}} X_0(z) \text{ with } n < 0$$

Solution of Difference Equations using Z-transform:

The difference equations can be easily solved using z-transform. Examples :

(1) Given that y(-1) = 5 and y(-2)=0, solve the difference equation y(n) - 3y(n-1) - 4y(n-2) = 0, $n \ge 0$. Solution : Consider the given difference equation

y(n) - 3y(n-1) - 4y(n-2) = 0

Taking unilateral z-transform of the given difference equation

 $Y(z) - 3[z^{-1}Y(z) + y(-1)] - 4[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)] = 0$

Put the initial conditions in above equation, we get

$$Y(z) - 3 [z^{-1}Y(z) + 5] - 4 [z^{-2}Y(z) + 5z^{-1} + 0] = 0$$

$$Y(z)[1 - 3z^{-1} - 4z^{-2}] - 20z^{-1} - 15 = 0$$

$$Y(z) = \frac{15 + 20z^{-1}}{1 - 3z^{-1} - 4z^{-2}} = \frac{z(15z + 20)}{z^2 - 3z - 4}$$

$$\frac{Y(z)}{z} = \frac{(15z+20)}{z^2 - 3z - 4} = \frac{(15z+20)}{(z+1)(z-4)}$$

Using partial faction method, the above equation can be written as,

$$\frac{(15z+20)}{(z+1)(z-4)} = \frac{A}{(z+1)} + \frac{B}{(z-4)}$$



$$(15z + 20) = A(z - 4) + B(z + 1)$$

∴
$$A|_{z=-1} = \frac{15z+20}{(z-4)} = -1$$
 and $B|_{z=4} = \frac{15z+20}{(z+1)} = 16$

$$\therefore \frac{Y(z)}{z} = \frac{-1}{(z+1)} + \frac{16}{(z-4)}$$

$$\therefore Y(z) = \frac{-z}{(z+1)} + \frac{16z}{(z-4)} = \frac{-1}{(1+z^{-1})} + \frac{-16}{(1-4z^{-1})}$$

 Taking Inverse Z-transform of the above equation, we get

$$y(n) = -(-1)^n u(n) + 16 (4)^n u(n) = [-(-1)^n + 16 (4)^n] u(n)$$

(2) Solve the difference equation using z-transform method x(n-2) - 9x(n-1) + 18x(n) = 0. Initial conditions are x(-1)=1, x(-2) =9.

Solution : Consider the given difference equation x(n-2) - 9x(n-1) + 18x(n) = 0

Taking unilateral z-transform of the given difference equation

$$[z^{-2}X(z) + z^{-1}x(-1) + x(-2)] -9[z^{-1}X(z) + x(-1)] +18 X(z) = 0$$

Put the initial conditions in above equation, we get

$$[z^{-2}X(z) + z^{-1} + 9] - 9[z^{-1}X(z) + 1] + 18 X(z) = 0$$

$$[z^{-2} - 9z^{-1} + 18] X(z) + z^{-1} = 0$$

$$X(z) = \frac{-z^{-1}}{z^{-2} - 9z^{-1} + 18} = \frac{-z}{1 - 9z + 18z^{2}}$$

$$X(z) = \frac{-1}{z^{-1}} = \frac{-z}{1 - 9z + 18z^{2}}$$

$$\frac{z}{z} = \frac{1}{18z^2 - 9z + 1} = \frac{1}{(6z - 1)(3z - 1)}$$

Using partial faction method, the above equation can be written as,

$$\frac{-1}{(6z-1)(3z-1)} = \frac{A}{(6z-1)} + \frac{B}{(3z-1)}$$

$$-1 = A(3z-1) + B(6z-1)$$

$$\therefore A|_{z=\frac{1}{6}} = \frac{-1}{(3z-1)} = \frac{1}{2} \quad \text{and} \quad B|_{z=\frac{1}{3}} = \frac{-1}{(6z-1)} = -1$$

$$\therefore \frac{X(z)}{z} = \frac{\frac{1}{2}}{(6z-1)} - \frac{1}{(3z-1)}$$
Now X(z) = $\frac{\frac{1}{2}z}{(6z-1)} - \frac{z}{(3z-1)} = \frac{\frac{1}{3}}{(1-\frac{1}{6}z^{-1})} - \frac{\frac{1}{3}}{(1-\frac{1}{3}z^{-1})}$
nverse Z-transform of the above equation, we get

Taking In

$$x(n) = \frac{1}{3} \left(\frac{1}{6}\right)^n u(n) - \frac{1}{3} \left(\frac{1}{3}\right)^n u(n) = \left[\frac{1}{3} \left(\frac{1}{6}\right)^n - \frac{1}{3} \left(\frac{1}{3}\right)^n\right] u(n)$$



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